TRINITY COLLEGE

Year 12 Mathematics Specialist Units 3, 4
Test 4 2020

Section 1 Calculator Free
Integration and Applications of Integration

STUDENT'S NAME ______________________________

DATE: Monday 27 July **TIME:** 33 minutes **MARKS**: 33

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser, Formula Booklet

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (4 marks)

Determine the following integrals:

(a) $\int \frac{x-1}{x}dx$ [2]

$= \int 1 - \frac{1}{x} \, dx$

$= x - \ln|x| + c$

(b) $\int x\cos(x^2)dx$ [2]

$= \frac{1}{2}\int 2x \cos x^2 \, dx$

$= \frac{1}{2}\sin x^2 + c$

2. (9 marks)

Determine the following integrals:

(a) $\displaystyle\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta$ [3]

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= -\frac{1}{2}\displaystyle\int \frac{-\sin\theta}{\cos\theta+1}\,d\theta$

$= -\frac{1}{2}\ln|\cos\theta+1| + c$

(b) $\displaystyle\int \cos^3 x\,dx$ [3]

$= \displaystyle\int \cos x(1-\sin^2 x)\,dx$

$= \displaystyle\int \cos x - \cos x\sin^2 x\,dx$

$= \sin x - \frac{\sin^3 x}{3} + c$

(c) $\displaystyle\int \frac{2x^2}{x+1}\,dx$ [3]

$= 2\displaystyle\int \frac{x^2}{x+1}\,dx$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2+0x+0 \\ & x^2+x \\ \hline & -x+0 \\ & -x-1 \\ \hline & 1 \end{array}$$

$= 2\displaystyle\int x-1+\frac{1}{x+1}\,dx$

$= 2\left[\frac{x^2}{2} - x + \ln|x+1|\right] + c$

$= x^2 - 2x + 2\ln|x+1| + c$

3. (6 marks)

(a) Express $\dfrac{\frac{7}{2}-x}{(x-1)(2x+3)}$ in the form $\dfrac{a}{x-1}+\dfrac{b}{2x+3}$. [3]

$$\Rightarrow \quad \frac{7}{2} - x = a(2x+3) + b(x-1)$$

$$\text{if } x=1 \quad \Rightarrow \quad \frac{5}{2} = 5a$$

$$\Rightarrow \quad a = \frac{1}{2}$$

$$\text{if } x = -\frac{3}{2} \quad \Rightarrow \quad 5 = b\left(-\frac{5}{2}\right)$$

$$b = -2$$

$$\therefore \quad \frac{\frac{7}{2}-x}{(x-1)(2x+3)} = \frac{1/2}{x-1} - \frac{2}{2x+3}$$

(b) Hence, determine $\displaystyle\int \frac{\frac{7}{2}-x}{(x-1)(2x+3)}\,dx$ [3]

$$= \int \frac{1/2}{x-1} - \frac{2}{2x+3} \quad dx$$

$$= \frac{1}{2}\ln|x-1| - \ln|2x+3| + c$$

4. (5 marks)

Evaluate exactly: $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$ using the substitution $x = 2\sin\theta$

$$= \int_0^{\pi/6} \frac{1}{\sqrt{4-4\sin^2\theta}}\, 2\cos\theta \, d\theta$$

$x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

when $x=1 \Rightarrow \frac{1}{2} = \sin\theta$

$\theta = \pi/6$

$x=0 \Rightarrow 0 = \sin\theta$

$\theta = 0$

$$= \int_0^{\pi/6} \frac{1}{2\cancel{\cos\theta}} \cdot 2\cancel{\cos\theta}\, d\theta$$

$$= \int_0^{\pi/6} d\theta$$

$$= \left[\theta\right]_0^{\pi/6}$$

$$= \pi/6$$

5. (9 marks)

Consider the integrals $I=\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$ and $J=\int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx$

(a) Use the substitution $u=a-x$ to show that $I=J$. [3]

$u=a-x$

$\frac{du}{dx}=-1$

when $x=a$ $u=0$

$x=0$ $u=a$

$$\therefore\ I=\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}\,(-du)$$

$$=\int_0^a \frac{f(a-u)}{f(u)+f(a-u)}\,du$$

$$=\int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx$$

$$=J$$

(b) By considering $I+J$, or otherwise, evaluate I in terms of a. [2]

$$I+J=\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx+\int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx$$

$$=\int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx$$

$$=\int_0^a dx$$

$$=a$$

And $I=J$

hence $2I=a$ or $I=\frac{a}{2}$

(c) Use the result from (b) and $\cos\theta = \sin(\frac{\pi}{2} - \theta)$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin(x + \frac{\pi}{4})}\, dx$. [4]

$$\sin\left(x+\frac{\pi}{4}\right) = \sin x \cos \pi/4 + \cos x \sin \pi/4$$
$$= \frac{1}{\sqrt{2}}(\sin x + \cos x)$$

$$\therefore \int_0^{\pi/2} \frac{\sin x}{\sin(x+\pi/4)}\, dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{2}\sin x}{\sin x + \cos x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/2} \frac{\sin x}{\sin x + \sin(\pi/2 - x)}\, dx$$

Now $f(x) = \sin x$, $a = \frac{\pi}{2}$ and $I = \frac{a}{2}$

$$\therefore \int_0^{\pi/2} \frac{\sin x}{\sin(x+\pi/4)}\, dx = \sqrt{2} \times \frac{1}{2} \times a$$
$$= \sqrt{2} \times \frac{1}{2} \times \frac{\pi}{2}$$
$$= \frac{\sqrt{2}\pi}{4}$$

TRINITY COLLEGE

Year 12 Mathematics Specialist Units 3, 4
Test 4 2020

Section 2 Calculator Assumed

Integration and Applications of Integration

STUDENT'S NAME ____

DATE: Monday 27 July **TIME:** 17 minutes **MARKS**: 16

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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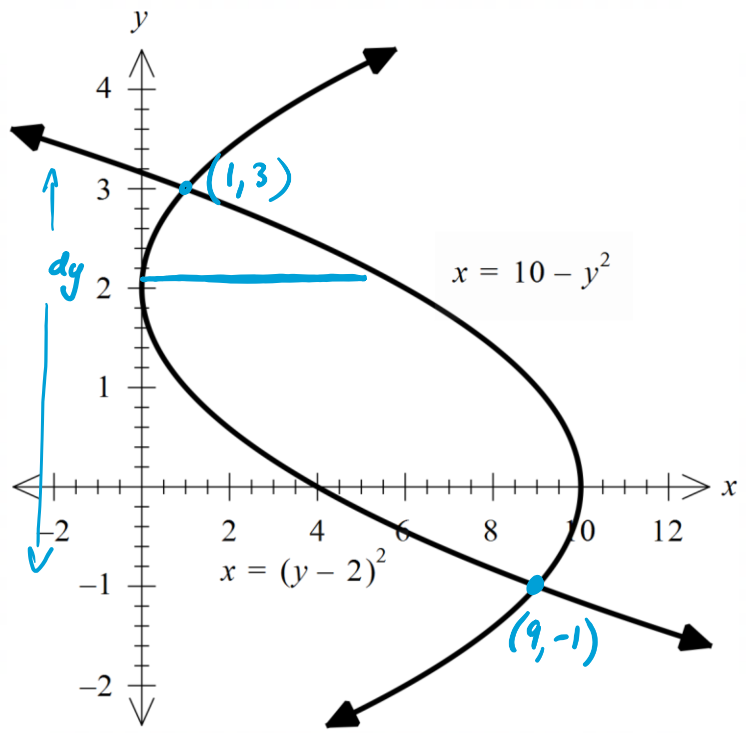
6. (2 marks)

Evaluate $\int_{-1}^{1} e^{-x^2}\,dx$ to 2 decimal places.

$= 1.49$

7. (8 marks)

Consider the two curves below.



(a) (i) Write an integral expression for the enclosed area between the curves. [2]

$$\text{Area} = \int_{-1}^{3} (10-y^2)-(y-2)^2 \, dy \quad \text{units}^2$$

(ii) Calculate the enclosed area. [2]

$$\text{Area} = 21\tfrac{1}{3} \quad \text{units}^2$$

(b) (i) Write down an integral expression for volume formed when the enclosed region is rotated about the y-axis. [2]

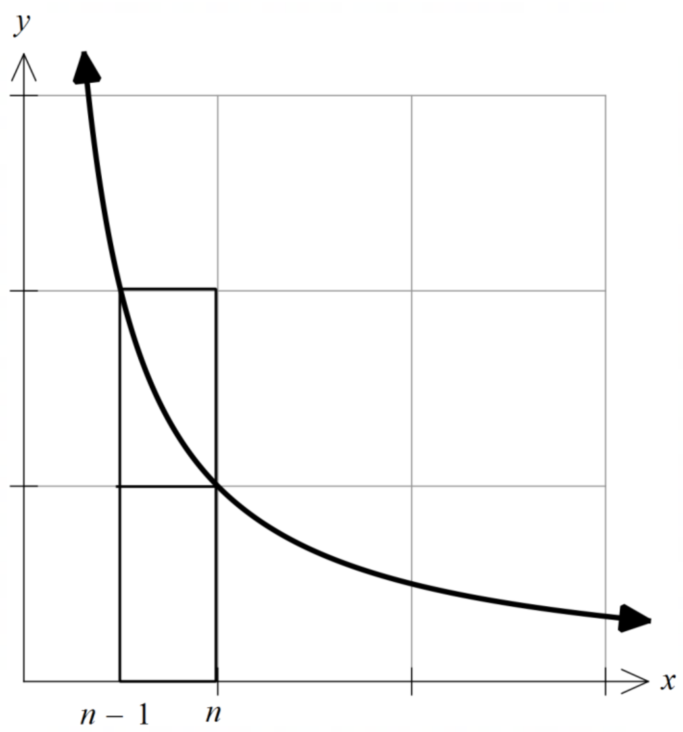
$$\text{Vol} = \pi \int_{-1}^{3} (10-y^2)^2-((y-2)^2)^2 \, dy \quad \text{units}^3$$

(ii) Calculate the volume formed when the enclosed region is rotated about the y-axis. [2]

$$\text{Vol} = 670.21 \quad \text{units}^3$$

8. (6 marks)

Let n be a positive integer greater than 1. The area of the region under the curve $y = \frac{1}{x}$ from $x = n-1$ to $x = n$ lies between the areas of the two rectangles, as shown in the diagram.



(a) Determine an expression for the area of the larger rectangle. [1]

Area $= l \times w$

$= \frac{1}{n-1} \times 1$

(b) Use the diagram to show that the area under the curve between $n-1$ and n satisfies $\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$ [2]

Smaller rect $< \int_{n-1}^{n} \frac{1}{x}\,dx <$ larger rect

$\frac{1}{n} < \left[\ln|x|\right]_{n-1}^{n} < \frac{1}{n-1}$

$\frac{1}{n} < \ln|n| - \ln|n-1| < \frac{1}{n-1}$

$\frac{1}{n} < \ln\frac{n}{n-1} < \frac{1}{n-1}$

(c) Use the result from (b) to show that $e^{\frac{-n}{(n-1)}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$ [3]

$$\frac{1}{n} < \ln\frac{n}{n-1} < \frac{1}{n-1}$$

$$e^{1/n} < \frac{n}{n-1} < e^{1/n-1}$$

$$\frac{1}{e^{1/n}} > \frac{n-1}{n} > \frac{1}{e^{1/n-1}}$$

$$\frac{1}{e^{1/n-1}} < \frac{n-1}{n} < \frac{1}{e^{1/n}}$$

$$\left(\frac{1}{e^{1/n-1}}\right)^n < \left(\frac{n-1}{n}\right)^n < \left(\frac{1}{e^{1/n}}\right)^n$$

$$\frac{1}{e^{n/n-1}} < \left(\frac{n}{n} - \frac{1}{n}\right)^n < \frac{1}{e^{n/n}}$$

$$e^{-n/(n-1)} < \left(1-\frac{1}{n}\right)^n < e^{-1}$$